

Nonrelativistic conformal groups and their dynamical realizations

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Abstract

Nonrelativistic conformal groups, indexed by $l = \frac{N}{2}$, are analyzed. Under the assumption that the mass parametrizing the central extension is nonvanishing the coadjoint orbits are classified and described in terms of convenient variables. It is shown that the corresponding dynamical system describes, within Ostrogradski framework, the nonrelativistic particle obeying $(N + 1)$ -th order equation of motion. As a special case, the Schrödinger group and the standard Newton equations are obtained for $N = 1$ ($l = \frac{1}{2}$).

1 Introduction

Historically, the structure which is now called the Schrödinger group has been discovered in XIX century in the context of classical mechanics [1] and heat equation [2]. It has been rediscovered in XX century as the maximal symmetry group of free motion in quantum mechanics [3]-[10]. Much attention has been paid to the structure of Schrödinger group and its geometrical status [7], [11]-[14].

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The Schrödinger group, when supplemented with space dilatation transformations becomes $l = \frac{1}{2}$ member of the whole family of nonrelativistic conformal groups [15, 16], indexed by halfinteger l . Various structural, geometric and physical aspects of the resulting Lie algebras have been intensively studied [17]-[35]. For $l = \frac{N}{2}$, N -odd (N -even in the case of dimension two), the nonrelativistic conformal algebra admits central extension. Then, as it has been shown in Ref. [28], it becomes the symmetry algebra of free non-relativistic particle obeying $(N + 1)$ -th order equation of motion.

In the present paper we use the orbit method [36]-[39] to construct the most general dynamical systems on which the nonrelativistic conformal groups act transitively as symmetries. We find that the basic variables are coordinates and momenta together with internal variables obeying $SU(2)$ commutation rules (in the sense of Poisson brackets) and underlying trivial dynamics; the remaining internal variables obey $SL(2, \mathbb{R})$ (or $SO(2, 1)$) commutation rules and equation of motion of conformal quantum mechanics [40] in global formulation [41].

All symmetry generators split into two parts: the external one constructed out of coordinates and momenta (like orbital angular momentum) and internal one (like spin). The symmetry transformations are implemented as canonical transformations.

The standard free dynamics is obtained by selecting the trivial orbit for $SL(2, \mathbb{R})$ variables.

The results heavily rely on the fact that the conformal algebras under considerations admit central extensions. For vanishing "mass" parameters (as well as for conformal algebras which do not admit central extension) the classification of orbits is more complicated and the physical interpretation in such cases remains slightly obscure.

2 The Schrödinger symmetry

We start with the $l = \frac{1}{2}$ Galilean conformal algebra (according to the terminology of Ref. [15, 16]). It consists of rotations \vec{J} , translations \vec{P} , boosts \vec{B} and time translations H which form the Galilean algebra, together with dilatations D , conformal transformations K and, finally, space dilatations

D_s . The nontrivial commutation rules read

$$\begin{aligned}
[J_i, J_k] &= i\epsilon_{ikl}J_l, & [J_i, P_k] &= i\epsilon_{ikl}P_l, & [J_i, B_k] &= i\epsilon_{ikl}B_l, \\
[B_i, H] &= iP_i, \\
[D, H] &= iH, & [D, K] &= -iK, & [K, H] &= 2iD, \\
[D, P_i] &= \frac{i}{2}P_i, & [D, B_i] &= \frac{-i}{2}B_i, & [K, P_i] &= iB_i, \\
[D_s, P_i] &= iP_i, & [D_s, B_i] &= iB_i.
\end{aligned} \tag{1}$$

Deleting D_s one obtains twelvedimensional Schrödinger algebra which admits, similarly to the Galilei algebra, central extension defined by additional nontrivial commutator

$$[B_i, P_k] = iM\delta_{ik}. \tag{2}$$

The structure of centrally extended Schrödinger algebra is well known. First, we have $su(2)$ (or $so(3)$) algebra spanned by J'_i s; furthermore, H , D and K span the conformal algebra which is isomorphic to $so(2, 1)$ (or $sl(2, \mathbb{R})$). To see this one defines

$$N^0 = \frac{1}{2}(H + K), \quad N^1 = \frac{1}{2}(K - H), \quad N^2 = D, \tag{3}$$

which yields

$$[N^\alpha, N^\beta] = i\epsilon^{\alpha\beta}_\gamma N^\gamma, \quad \alpha, \beta, \gamma = 0, 1, 2; \tag{4}$$

where $\epsilon^{012} = \epsilon_{012} = 1$, and $g_{\mu\nu} = \text{diag}(+, -, -)$. Therefore \vec{J}, H, K and D span direct sum $su(2) \oplus so(2, 1)$. Finally, \vec{P}, \vec{B} and M form a nilpotent algebra which, at the same time, carries a representation of $su(2) \oplus so(2, 1)$. To express this fact in compact way we define the spinor representation of $so(2, 1)$:

$$\tilde{N}^0 = \frac{1}{2}\sigma_2, \quad \tilde{N}^1 = \frac{i}{2}\sigma_1, \quad \tilde{N}^2 = \frac{i}{2}\sigma_3. \tag{5}$$

Moreover, denoting $X_{1i} = P_i$, $X_{2i} = B_i$ one finds simple form of the action of $su(2) \oplus so(2, 1)$ on the space spanned by \vec{P}, \vec{B} and M

$$[J_i, X_{ak}] = i\epsilon_{ikl}X_{al}, \quad [N^\alpha, X_{ai}] = X_{bi}(\tilde{N}^\alpha)_{ba}, \tag{6}$$

where $a, b = 1, 2$. The commutation rule (2) takes the form

$$[X_{ai}, X_{bj}] = -iM\epsilon_{ab}\delta_{ij}. \tag{7}$$

The matrices \tilde{N}^α are all purely imaginary and span the defining representation of $sl(2, \mathbb{R})$. In fact, the group $SL(2, \mathbb{R})$ is nothing but the group $Spin(2, 1)^+$. The Schrödinger algebra can be thus integrated to the group $S = (SU(2) \times SL(2, \mathbb{R})) \ltimes R_7$, where R_7 is seven-dimensional nilpotent group (topologically isomorphic to \mathbb{R}^7) and the semidirect product is defined by the $D^{(1, \frac{1}{2})} \oplus D^{(0, 0)}$ representation of $SU(2) \times SL(2, \mathbb{R})$.

Let us consider the coadjoint action of Schrödinger group S . Denote the dual basis elements by $\vec{J}, \vec{P}, \vec{B}$ etc. The general element of the dual space to the Lie algebra of S is written as

$$X = \vec{J} \vec{J} + \vec{\xi} \vec{P} + \vec{\zeta} \vec{B} + h \tilde{H} + d \tilde{D} + k \tilde{K} + m \tilde{M}. \quad (8)$$

Having characterized the global structure of S we could consider the full action of S on X . However, for our purposes it is sufficient to compute the coadjoint action of one-parameter subgroups generated by the basic elements of the Lie algebra. The results are summarized in Table 1 below.

Table 1: Coadjoint action of S .

$Ad_g^* \backslash g$	$e^{i\vec{a}\vec{P}}$	$e^{i\vec{v}\vec{B}}$	$e^{-i\tau H}$	$e^{i\lambda D}$	e^{iuK}	$e^{i\vec{\omega}\vec{J}}$
\vec{J}'	$\vec{J} - \vec{a} \times \vec{\xi}$	$\vec{J} - \vec{v} \times \vec{\zeta}$	\vec{J}	\vec{J}	\vec{J}	\vec{R}_J
$\vec{\xi}'$	$\vec{\xi}$	$\vec{\xi} + m\vec{v}$	$\vec{\xi}$	$e^{\frac{\lambda}{2}} \vec{\xi}$	$\vec{\xi} + u\vec{\zeta}$	\vec{R}_ξ
$\vec{\zeta}'$	$\vec{\zeta} - m\vec{a}$	$\vec{\zeta}$	$\vec{\zeta} + \tau\vec{\xi}$	$e^{-\frac{\lambda}{2}} \vec{\zeta}$	$\vec{\zeta}$	\vec{R}_ζ
h'	h	$h + \frac{m\vec{v}^2}{2} + \vec{v}\vec{\xi}$	h	$e^\lambda h$	$h + 2ud + u^2k$	h
d'	$d - \frac{1}{2}\vec{a}\vec{\xi}$	$d + \frac{1}{2}\vec{v}\vec{\zeta}$	$d + \tau h$	d	$d + uk$	d
k'	$k - \vec{a}\vec{\zeta} + \frac{1}{2}m\vec{a}^2$	k	$k + 2\tau d + \tau^2 h$	$e^{-\lambda} k$	k	k
m'	m	m	m	m	m	m

here $(\vec{R}_J)_k = R_{kl} j_l$ etc.

In order to find the structure of coadjoint orbits note that m is invariant under the coadjoint action of S . In what follows we assume that $m > 0$ (in fact, it is sufficient to take $m \neq 0$). Once this assumption is made, the classification of orbits become quite simple. Using the results collected in Table 1 we conclude that each orbit contains the point corresponding to $\vec{\xi} = 0, \vec{\zeta} = 0$. Moreover, the stability subgroup of the submanifold $\vec{\xi} = 0, \vec{\zeta} = 0$ is $SU(2) \times SL(2, \mathbb{R}) \times \mathbb{R}$ where the last factor is the subgroup generated by M and can be neglected. The orbits of $SU(2) \times SL(2, \mathbb{R})$ are the products

of orbits of both factors. For $SU(2)$ any coadjoint orbit is a 2-sphere (or a point) which can be parametrized by vector \vec{s} of fixed length, $\vec{s}^2 = s^2$. To describe the orbits of $SL(2, \mathbb{R})$ (which is equivalent, as far as coadjoint action is concerned, to $SO(2, 1)$) we define, in analogy with eq. (3),

$$\chi^0 = \frac{1}{2}(h + k), \quad \chi^1 = \frac{1}{2}(-h + k), \quad \chi^2 = d. \quad (9)$$

Then, by standard arguments, the full list of orbits reads:

$$\begin{aligned} \mathcal{H}_\sigma^+ &= \{\chi^\mu : g_{\mu\nu} \chi^\mu \chi^\nu = \sigma^2, \chi^0 > 0\}, \\ \mathcal{H}_\sigma^- &= \{\chi^\mu : g_{\mu\nu} \chi^\mu \chi^\nu = \sigma^2, \chi^0 < 0\}, \\ \mathcal{H}_0^+ &= \{\chi^\mu : g_{\mu\nu} \chi^\mu \chi^\nu = 0, \chi^0 > 0\}, \\ \mathcal{H}_0^- &= \{\chi^\mu : g_{\mu\nu} \chi^\mu \chi^\nu = 0, \chi^0 < 0\}, \\ \mathcal{H}_\sigma &= \{\chi^\mu : g_{\mu\nu} \chi^\mu \chi^\nu = -\sigma^2\}, \\ \mathcal{H}_0 &= \{0\}. \end{aligned} \quad (10)$$

Consequently, any coadjoint orbit of S (with nonvanishing m) contains the point

$$\vec{s}\tilde{J} + (\chi^0 - \chi^1)\tilde{H} + \chi^2\tilde{D} + (\chi^0 + \chi^1)\tilde{K} + m\tilde{M}, \quad (11)$$

where $\vec{s} \in S^2$ and χ^μ is a point on one of the manifolds \mathcal{H} listed above. We see that any orbit is characterized by the values of m, \vec{s}^2, χ^2 and, for $\chi^2 \geq 0$, the sign of χ^0 . Let us note that the above invariants correspond to the Casimir operators of Schrödinger algebra

$$\begin{aligned} C_1 &= M, \\ C_2 &= (M\vec{J} - \vec{B} \times \vec{P})^2, \\ C_3 &= \left(MH - \frac{\vec{P}^2}{2}\right) \left(MK - \frac{\vec{B}^2}{2}\right) + \left(MK - \frac{\vec{B}^2}{2}\right) \left(MH - \frac{\vec{P}^2}{2}\right) + \\ &\quad - 2 \left(MD - \frac{\vec{B}\vec{P}}{4} - \frac{\vec{P}\vec{B}}{4}\right)^2. \end{aligned} \quad (12)$$

The whole coadjoint orbit of S can be obtained by applying $g(\vec{a})$ and $g(\vec{v})$ to all points (11) with \vec{s} and χ^μ varying over their orbits. Calling $\vec{a} = -\vec{x}$ and

$\vec{v} = \vec{p}/m$ one finds the following parametrization of coadjoint orbits

$$\begin{aligned}
\vec{j} &= \vec{x} \times \vec{p} + \vec{s}, \\
\vec{\xi} &= \vec{p}, \\
\vec{\zeta} &= m\vec{x}, \\
h &= \frac{\vec{p}^2}{2m} + \chi^0 - \chi^1, \\
d &= \frac{1}{2}\vec{x}\vec{p} + \chi^2, \\
k &= \frac{m}{2}\vec{x}^2 + \chi^0 + \chi^1.
\end{aligned} \tag{13}$$

We see that the phase-space variables are $\vec{x}, \vec{p}, \vec{s}$ and χ^μ . The Poisson brackets implied by Kirillov symplectic structure read

$$\begin{aligned}
\{x_i, p_k\} &= \delta_{ik}, \\
\{s_i, s_k\} &= \epsilon_{ikl} s_l, \\
\{\chi^\alpha, \chi^\beta\} &= \epsilon^{\alpha\beta}{}_\gamma \chi^\gamma,
\end{aligned} \tag{14}$$

while the corresponding equations of motion take the form

$$\begin{aligned}
\dot{\vec{x}} &= \frac{\vec{p}}{m}, \quad \dot{\vec{p}} = 0, \quad \dot{\vec{s}} = 0, \\
\dot{\chi}^0 &= \chi^2, \quad \dot{\chi}^1 = \chi^2, \quad \dot{\chi}^2 = -\chi^1 + \chi^0.
\end{aligned} \tag{15}$$

We can summarize our findings. The tendimensional orbits are parametrized by $\vec{x}, \vec{p}, \vec{s}$ and χ^μ subject to the constraints $\vec{s}^2 = \text{const.}$ and $g_{\mu\nu} \chi^\mu \chi^\nu = \text{const.}$ and equipped with the symplectic structure defined by eqs. (14) and dynamics given by eqs. (15).

One can say that, besides the standard canonical variables \vec{x} and \vec{p} , there are two kinds of "internal" degrees of freedom – ordinary spin variables \vec{s} and $SO(2, 1)$ "spin" degrees of freedom χ^μ . Note that, contrary to the true spin variables, χ^μ have nontrivial dynamics.

3 Special cases

Making the trivial choice $\mathcal{H}_0 = \{0\}$ of the $SL(2, \mathbb{R})$ orbit one finds the standard realization of Schrödinger group as the symmetry of free dynamics.

The structure of the phase space is the same as in the case of Galilei group except that the internal energy (the Casimir of Galilei group) vanishes. The additional generators K and D are constructed as the elements of enveloping algebra of Galilei algebra.

The Schrödinger algebra contains also Newton-Hooke algebra as subalgebra. This is easily seen by redefining the Hamiltonian: $H \rightarrow H \pm \omega^2 K$. The Galilei and Newton-Hooke algebras are not isomorphic. However, due to the fact that, in the special case under consideration, K belongs to the enveloping algebra of Galilei one, Newton-Hooke algebra is contained in this enveloping algebra and reverse.

In the general case of arbitrary orbit of $SL(2, \mathbb{R})$ both Galilei and Newton-Hooke algebras/groups do not act transitively. However, one can reduce the phase space by abandoning the variables χ^μ except the combination $\chi^0 - \chi^1$ $((1 + \omega^2)\chi^0 + (-1 + \omega^2)\chi^1)$ which is now viewed as a constant representing the value of internal energy for Galilei (Newton-Hooke) algebra. The reduced phase space coincides with the one obtained by applying the orbit method directly to the Galilei or Newton-Hooke groups.

4 Canonical transformations

From the basic functions (13) one can construct the generators (in the sense of canonical formalism) of group transformations. Due to the fact that the Hamiltonian is an element of the Lie algebra of symmetry group the symmetry generators depend, in general, explicitly on time. To construct the explicitly time dependent generators of symmetry transformation one notes that the dynamics induces an internal automorphism of Lie algebra of Schrödinger group. Therefore, the relevant generators (providing the integrals of motion which existence is implied by the symmetry under consideration) are obtained by inverting this automorphism. The result reads

$$\begin{aligned} j_k &= j_k(t), \quad p_k = p_k(t), \\ x_k &= x_k(t) - \frac{t}{m} p_k(t), \quad h = h(t), \\ k &= k(t) - 2td(t) + t^2 h(t), \quad d = d(t) - th(t). \end{aligned} \tag{16}$$

In order to find the transformation generated by left-hand sides of eqs. (16) let us note that the one-parameter group of symmetry transformations of

canonical variables η

$$\begin{aligned} t' &= g_1(t; c) \simeq t + \delta c \tilde{g}_1(t), \\ \eta'(t') &= g_2(\eta(t), t; c) \simeq \eta(t) + \delta c \tilde{g}_2(\eta(t), t), \end{aligned} \quad (17)$$

is related to its canonical generator $G(t)$ via

$$\delta_0 \eta = \delta c \{ \eta, G \}, \quad (18)$$

where

$$\delta_0 \eta = \eta'(t) - \eta(t) = \delta c (\tilde{g}_2(\eta(t), t) - \dot{\eta}(t) \tilde{g}_1(t)). \quad (19)$$

As an example consider the transformation generated by k . By comparing eq. (16) for k and eq. (19) we find

$$\tilde{g}_1(t) = -t^2. \quad (20)$$

Integration of eq. (20) gives

$$t' = \frac{t}{1 + ct}. \quad (21)$$

Having described the transformation properties of time variable one determines that of x_i and p_i . To this end it is convenient to use the simplified form of k , $k_s = k(t) - 2td(t)$ together with the replacement $t \rightarrow t/(1 + ct)$:

$$\frac{dx_i}{dc} = \{x_i, k - \frac{2t}{1 + ct}d\} = -\frac{t}{1 + ct}x_i, \quad (22)$$

yielding

$$x'_i = \frac{x_i}{1 + ct}. \quad (23)$$

Analogously

$$p'_i = p_i(1 + ct) - mcx_i. \quad (24)$$

Similarly, one can consider the action of conformal transformation on "internal" variables.

The action of conformal transformation on time variable, eq. (21), can be extended to the whole Schrödinger group. In fact, by deleting the Hamiltonian H one obtains the subgroup of S . Therefore, it is possible to define the nonlinear action of Schrödinger group on onedimensional coset space. It is singular (cf. eq. (21)) if one uses exponential parametrization because the latter provides only local map. Taking into account global topology requires more care [23]. The action of other generators may be described in a similar way.

5 N-Galilean Conformal Symmetry

Higher dimensional nonrelativistic conformal algebras are constructed according to the following unique scheme. One takes the direct sum $su(2) \oplus sl(2, \mathbb{R}) \oplus \mathbb{R}$, where the last term corresponds to the spatial dilatation D_s . This is supplemented by $3(N+1)$ Abelian algebra (here $l = N/2$) which carries the $D^{(1, \frac{N}{2})}$ representation of $SU(2) \otimes SL(2, \mathbb{R})$; moreover, all new generators correspond to the eigenvalue 1 of D_s . Call $\vec{C}_i = (C_i^a, \quad a = 1, 2, 3)$, $i = 0, 1, \dots, N$, the new generators. The relevant commutation rules involving \vec{C}_i read

$$\begin{aligned} [D_s, C_j^a] &= iC_j^a, \\ [J^a, C_j^b] &= i\epsilon_{abd}C_j^d, \\ [H, C_j^a] &= -ijC_{j-1}^a, \\ [D, C_j^a] &= i\left(\frac{N}{2} - j\right)C_j^a, \\ [K, C_j^a] &= i(N - j)C_{j+1}^a. \end{aligned} \tag{25}$$

As previously we delete the space dilatation operator D_s and consider the question of the existence of central extension of the Abelian algebra spanned by \vec{C}'_i s. To solve it one can consider the relevant Jacobi identities or analyze the transformation properties under $SU(2) \times SL(2, \mathbb{R})$. The second order $SU(2)$ invariant tensor, i.e. Kronecker delta δ^{ab} in arbitrary dimension (and tensor ϵ^{ab} for dimension two), is symmetric (antisymmetric, respectively), so the existence of central extension is equivalent to the existence antisymmetric (symmetric) $SL(2, \mathbb{R})$ invariant tensor. Taking into account that $N+1$ -dimensional irreducible representations of $SL(2, \mathbb{R})$ may be obtained from symmetrized tensor product of N basic representation one easily concludes that an invariant antisymmetric (symmetric) tensor exists only for N odd (for N even in the case dimension two) (see Ref. [42]).

5.1 N-odd

In this case the relevant central extension reads [28]

$$[C_j^a, C_k^b] = i\delta^{ab}\delta^{N,j+k}(-1)^{\frac{k-j+1}{2}}k!j!M, \tag{26}$$

for $j, k = 0, 1, \dots, N$ and $a, b = 1, 2, 3$. In order to classify the coadjoint orbits we put, in analogy to eq. (8),

$$X = \vec{j}\tilde{J} + \vec{c}_i\tilde{C}_i + h\tilde{H} + d\tilde{D} + k\tilde{K} + m\tilde{M}. \quad (27)$$

Again, m is invariant under the coadjoint action; we assume that $m > 0$. Consider the coadjoint action of $\exp(ix_k^a C_k^a)$. It reads

$$\begin{aligned} m' &= m, \\ j'^b &= j^b - \epsilon_{bad} \sum_{j=0}^N x_j^a c_j^d - \frac{m}{2} \sum_{j=0}^N (-1)^{j-\frac{N+1}{2}} \epsilon_{bca} x_j^a x_{N-j}^c j!(N-j)!, \\ c_j'^b &= c_j^b + (-1)^{j-\frac{N-1}{2}} m j!(N-j)! x_{N-j}^b, \\ h' &= h + \sum_{j=0}^{N-1} (j+1) x_{j+1}^b c_j^b + \frac{m}{2} \sum_{j=1}^N (-1)^{j-\frac{N+1}{2}} j!(N-j+1)! x_j^a x_{N-j+1}^a, \\ d' &= d - \sum_{j=0}^N \left(\frac{N}{2} - j\right) x_j^b c_j^b + \frac{m}{2} \sum_{j=0}^N \left(\frac{N}{2} - j\right) (-1)^{j-\frac{N+1}{2}} j!(N-j)! x_j^a x_{N-j}^a, \\ k' &= k - \sum_{j=1}^N (N-j+1) x_{j-1}^b c_j^b + \frac{m}{2} \sum_{j=0}^{N-1} (-1)^{j-\frac{N-1}{2}} (j+1)!(N-j)! x_j^a x_{N-j-1}^a, \end{aligned} \quad (28)$$

We see that, as in the case of Schrödinger group, any orbit contains the points

$$\vec{s}\tilde{J} + (\chi^0 - \chi^1)\tilde{H} + \chi^2\tilde{D} + (\chi^0 + \chi^1)\tilde{K} + m\tilde{M}, \quad (29)$$

where, again, $\vec{s} \in S^2$ and χ^μ belongs to one of the orbits (10). The whole orbit is produced by acting with $\exp(ix_k^a C_k^a)$ on the above points. As a result

we arrive at the following parametrization

$$\begin{aligned}
j^b &= s^b - \frac{m}{2} \sum_{j=0}^N (-1)^{j-\frac{N+1}{2}} \epsilon_{bcd} x_j^a x_{N-j}^c j!(N-j)!, \\
c_j^b &= (-1)^{j-\frac{N-1}{2}} m j!(N-j)! x_{N-j}^b, \\
h &= \chi^0 - \chi^1 + \frac{m}{2} \sum_{j=1}^N (-1)^{j-\frac{N+1}{2}} j!(N-j+1)! x_j^a x_{N-j+1}^a, \\
d &= \chi^2 + \frac{m}{2} \sum_{j=0}^N \left(\frac{N}{2} - j\right) (-1)^{j-\frac{N+1}{2}} j!(N-j)! x_j^a x_{N-j}^a, \\
k &= \chi^0 + \chi^1 + \frac{m}{2} \sum_{j=0}^{N-1} (-1)^{j-\frac{N-1}{2}} (j+1)!(N-j)! x_j^a x_{N-j-1}^a.
\end{aligned} \tag{30}$$

The invariants \vec{s}^2 and $g_{\mu\nu} \chi^\mu \chi^\nu$, which characterize the orbits, correspond to the Casimir operators

$$\begin{aligned}
C_1 &= M, \\
C_2 &= \left(M \vec{J} - \frac{1}{2} \sum_{j=0}^N \frac{(-1)^{j-\frac{N+1}{2}}}{j!(N-j)!} \vec{C}_j \times \vec{C}_{N-j} \right)^2, \\
C_3 &= (MH - A)(MK - B) + (MK - B)(MH - A) - 2(MD - C)^2,
\end{aligned} \tag{31}$$

where

$$\begin{aligned}
A &= \frac{1}{2} \sum_{j=1}^N \frac{(-1)^{j-\frac{N+1}{2}}}{(j-1)!(N-j)!} \vec{C}_{j-1} \vec{C}_{N-j}, \\
B &= -\frac{1}{2} \sum_{j=0}^{N-1} \frac{(-1)^{j-\frac{N+1}{2}}}{j!(N-j-1)!} \vec{C}_{j+1} \vec{C}_{N-j}, \\
C &= \frac{1}{2} \sum_{j=0}^N \frac{(-1)^{j-\frac{N+1}{2}}}{j!(N-j)!} \left(j - \frac{N}{2}\right) \vec{C}_j \vec{C}_{N-j}.
\end{aligned} \tag{32}$$

The basic dynamical variables are χ^μ , s^a and x_j . The Poisson bracket resulting from Kirillov symplectic structure reads

$$\{c_j^a, c_k^b\} = \delta^{ab} \delta^{N,j+k} (-1)^{\frac{k-j+1}{2}} k! j! m, \tag{33}$$

and implies

$$\{x_k^a, x_{N-k}^b\} = \frac{\delta^{ab}(-1)^{k-\frac{N+1}{2}}}{mk!(N-k)!}, \quad k = 0, 1, \dots, N. \quad (34)$$

It is easy to define Darboux coordinates for "external" variables. They read

$$\begin{aligned} x_k^a &= \frac{(-1)^{k-\frac{N+1}{2}}}{k!} q_k^a, \\ x_{N-k}^a &= \frac{1}{m(N-k)!} p_k^a, \end{aligned} \quad (35)$$

for $k = 0, \dots, \frac{N-1}{2}$, yielding the standard form of Poisson brackets

$$\{q_k^a, p_l^b\} = \delta^{ab} \delta_{kl}. \quad (36)$$

In terms of new variables the remaining one read

$$\begin{aligned} h &= \chi^0 - \chi^1 + \frac{1}{2m} \vec{p}_{\frac{N-1}{2}} \vec{p}_{\frac{N-1}{2}} + \sum_{k=1}^{\frac{N-1}{2}} \vec{q}_k \vec{p}_{k-1}, \\ d &= \chi^2 + \sum_{k=0}^{\frac{N-1}{2}} \left(\frac{N}{2} - k\right) \vec{q}_k \vec{p}_k, \\ k &= \chi^0 + \chi^1 + \frac{m}{2} \left(\frac{N+1}{2}\right)^2 \vec{q}_{\frac{N-1}{2}} \vec{q}_{\frac{N-1}{2}} - \sum_{k=0}^{\frac{N-3}{2}} (N-k)(k+1) \vec{q}_k \vec{p}_{k+1}, \\ \vec{j} &= \vec{s} + \sum_{k=0}^{\frac{N-1}{2}} \vec{q}_k \times \vec{p}_k. \end{aligned} \quad (37)$$

The above findings can be compared with those of Ref. [28]. In particular, the Hamiltonian h is the sum of two terms depending on "internal" ($sl(2, \mathbb{R})$) and "external" variables. The external part coincides with the Ostrogradski Hamiltonian [43] corresponding to the Lagrangian

$$L = \frac{m}{2} \left(\frac{d^{\frac{N+1}{2}} \vec{q}}{dt^{\frac{N+1}{2}}} \right)^2. \quad (38)$$

This can be easily seen by writing out the canonical equations of motion

$$\begin{aligned}\dot{\vec{q}}_k &= \vec{q}_{k+1}, \quad k = 0, \dots, \frac{N-3}{2}, \\ \dot{\vec{p}}_k &= -\vec{p}_{k-1}, \quad k = 1, \dots, \frac{N-1}{2} \\ \dot{\vec{q}}_{\frac{N-1}{2}} &= \frac{1}{m} \vec{p}_{\frac{N-1}{2}}, \quad \dot{\vec{p}}_0 = 0\end{aligned}\tag{39}$$

which, for the basic variable $\vec{q} = \vec{q}_0$, imply $\vec{q}^{(N+1)} = 0$.

5.2 N-even

As we have mentioned, in the case of dimension 2 for even N , there exists also the central extension of the Abelian algebra spanned by \vec{C} 's. The relevant commutators read:

$$[C_j^a, C_k^b] = -i\epsilon^{ab}\delta^{N,j+k}(-1)^{\frac{j-k}{2}}k!j!M,\tag{40}$$

where $a, b = 1, 2$, $j, k = 0, 1, \dots, N$. Let us take an arbitrary element X of dual space to the Lie algebra

$$X = j\tilde{J} + \vec{c}_i\tilde{\vec{C}}_i + h\tilde{H} + d\tilde{D} + k\tilde{K} + m\tilde{M}.\tag{41}$$

As previously, m is invariant under the coadjoint action; we can assume that $m > 0$. Consider the coadjoint action of $\exp(ix_k^a C_k^a)$. It reads

$$\begin{aligned}m' &= m, \\ j' &= j - \epsilon^{ba} \sum_{j=0}^N x_j^b c_j^a + \frac{m}{2} \sum_{j=0}^N (-1)^{\frac{2j-N}{2}} \epsilon^{ad} \epsilon^{bd} x_j^b x_{N-j}^a j!(N-j)!, \\ c_j'^b &= c_j^b - (-1)^{\frac{N-2j}{2}} m j!(N-j)! \epsilon^{ab} x_{N-j}^a, \\ h' &= h + \sum_{j=0}^{N-1} (j+1) x_{j+1}^b c_j^b + \frac{m}{2} \sum_{j=1}^N (-1)^{\frac{2j-N}{2}} j!(N-j+1)! \epsilon^{ab} x_j^b x_{N-j+1}^a, \\ d' &= d - \sum_{j=0}^N \left(\frac{N}{2} - j\right) x_j^b c_j^b - \frac{m}{2} \sum_{j=0}^N \left(-\frac{N}{2} + j\right) (-1)^{\frac{2j-N}{2}} j!(N-j)! \epsilon^{ab} x_j^b x_{N-j}^a, \\ k' &= k - \sum_{j=0}^{N-1} (N-j) x_j^b c_{j+1}^b - \frac{m}{2} \sum_{j=0}^{N-1} (-1)^{\frac{2j-N}{2}} (j+1)!(N-j)! \epsilon^{ba} x_j^b x_{N-j-1}^a.\end{aligned}\tag{42}$$

We see that, similarly to the case of N -odd, any orbit contains the points

$$s\tilde{J} + (\chi^0 - \chi^1)\tilde{H} + \chi^2\tilde{D} + (\chi^0 + \chi^1)\tilde{K} + m\tilde{M}, \quad (43)$$

where $s \in \mathbb{R}$ and χ^μ belongs to one of the orbits (10). Moreover, the whole orbit is produced by acting with $\exp(ix_k^a C_k^a)$ on the above points. Consequently, we have the following parametrization

$$\begin{aligned} j &= s + \frac{m}{2} \sum_{j=0}^N (-1)^{\frac{2j-N}{2}} \epsilon^{ad} \epsilon^{bd} x_j^b x_{N-j}^a j!(N-j)!, \\ c_j^b &= (-1)^{\frac{N-2j}{2}} m j!(N-j)! \epsilon^{ba} x_{N-j}^a, \\ h &= \chi^0 - \chi^1 + \frac{m}{2} \sum_{j=1}^N (-1)^{\frac{2j-N}{2}} j!(N-j+1)! \epsilon^{ab} x_j^b x_{N-j+1}^a, \\ d &= \chi^2 - \frac{m}{2} \sum_{j=0}^N \left(-\frac{N}{2} + j\right) (-1)^{\frac{2j-N}{2}} j!(N-j)! \epsilon^{ab} x_j^b x_{N-j}^a, \\ k &= \chi^0 + \chi^1 - \frac{m}{2} \sum_{j=0}^{N-1} (-1)^{\frac{2j-N}{2}} (j+1)!(N-j)! \epsilon^{ab} x_j^b x_{N-j-1}^a. \end{aligned} \quad (44)$$

By direct, but rather tedious, computations we check that the corresponding Casimir operators are of the form

$$\begin{aligned} C_1 &= M, \\ C_2 &= MJ - \frac{1}{2} \sum_{j=0}^N \frac{(-1)^{\frac{2j-N}{2}}}{j!(N-j)!} C_{N-j}^a C_j^a, \\ C_3 &= (MH - A)(MK - B) + (MK - B)(MH - A) - 2(MD - C)^2, \end{aligned} \quad (45)$$

where

$$\begin{aligned} A &= \frac{1}{2} \sum_{j=1}^N \frac{(-1)^{\frac{2j-N}{2}}}{(j-1)!(N-j)!} \epsilon^{ab} C_{j-1}^b C_{N-j}^a, \\ B &= -\frac{1}{2} \sum_{j=0}^{N-1} \frac{(-1)^{\frac{2j-N}{2}}}{j!(N-j-1)!} \epsilon^{ab} C_{j+1}^b C_{N-j}^a, \\ C &= \frac{1}{2} \sum_{j=0}^N \frac{(-1)^{\frac{2j-N}{2}}}{j!(N-j)!} \left(j - \frac{N}{2}\right) \epsilon^{ab} C_j^b C_{N-j}^a. \end{aligned} \quad (46)$$

The induced Poisson brackets of \vec{C} 's take the form

$$\{c_j^a, c_k^b\} = -\epsilon^{ab} \delta^{N,j+k} (-1)^{\frac{k-j}{2}} k! j! m, \quad (47)$$

(for χ^μ see eq. (14)). Now let us define new coordinates as follows

$$\begin{aligned} x_j^a &= \frac{(-1)^{\frac{N-2j}{2}}}{j!} q_j^a, \quad j = 0, \dots, \frac{N}{2} - 1, \quad a, b = 1, 2; \\ x_{N-j}^a &= \frac{1}{m(N-j)!} p_j^a, \quad j = 0, \dots, \frac{N}{2}, \quad a, b = 1, 2. \end{aligned} \quad (48)$$

Then the nonvanishing Poisson brackets read

$$\begin{aligned} \{q_j^a, p_j^b\} &= \delta^{ab} \delta_{jk}, \quad j, k = 0, \dots, \frac{N}{2} - 1, \quad a, b = 1, 2; \\ \{q_{\frac{N}{2}}^a, q_{\frac{N}{2}}^b\} &= \frac{1}{m} \epsilon^{ba}, \quad a, b = 1, 2. \end{aligned} \quad (49)$$

Let us introduce auxiliary notation (see eq. (32) in Ref. [28])

$$p_{\frac{N}{2}}^a = \frac{m}{2} \epsilon^{ba} q_{\frac{N}{2}}^b. \quad (50)$$

Then, the remaining dynamical variables take form

$$\begin{aligned} h &= \chi^0 - \chi^1 + \sum_{k=0}^{\frac{N}{2}-1} \vec{p}_k \vec{q}_{k+1}, \\ d &= \chi^2 + \sum_{k=0}^{\frac{N}{2}-1} \left(\frac{N}{2} - k \right) \vec{p}_k \vec{q}_k, \\ k &= \chi^0 + \chi^1 - \sum_{k=1}^{\frac{N}{2}-1} (N - k + 1) k \vec{p}_k \vec{q}_{k-1} - N \left(\frac{N}{2} + 1 \right) \vec{q}_{\frac{N}{2}-1} \vec{p}_{\frac{N}{2}}, \\ j &= s + \sum_{k=0}^{\frac{N}{2}} \vec{q}_k \times \vec{p}_k. \end{aligned} \quad (51)$$

These results, in the case of trivial orbit \mathcal{H}_0 , agree with the ones obtained in Ref. [28].

We conclude that the general dynamical system admitting N -Galilean conformal symmetry with N -odd (N -even in dimension two) as the symmetry group acting transitively is described by the "external" variables corresponding to higher derivative Lagrangian and two kinds of internal ones: spin variables \vec{s} (s , respectively) with trivial dynamics and $SL(2, \mathbb{R})$ spin variables χ^μ with nontrivial conformal invariant one. As in the case of Schrödinger algebra it is easy to construct the explicitly time-dependent integrals of motion. They generate the relevant symmetry transformations.

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